Numerical Experiments in Error Control for Sound Propagation Using a Damping Layer Boundary Treatment

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Acoustic propagation boundary treatment error control.

- Propagation by the Linearized Euler Equations (LEE).

- Algorithms are 3^{rd}, 5^{th}, 7^{th} and 9^{th} order in time and space.

- Time accurate Damping Layer (DL) boundary treatment.

- Polynomial damping profiles are 2^{nd}, 4^{th}, 6^{th} and 8^{th} order.

- Propagating solution uniformly set to zero on outer boundary.

- Maximum relative absolute errors from $O[10^{-3}]$ to $O[10^{-7}]$.

Linearized Euler Equation Conventions.

- Cartesian coordinates (x, y) in two space dimensions.

- Primitive variables $\vec{V} = (\sigma, u, v, p)^T$, with $\sigma = 1/\rho$.

- Nondimensionalized with ρ_R, L_R and speed of sound a_R.

- $\sigma_R = 1/\rho_R$, $t_R = L_R/a_R$, $p_R = a_R^2 \rho_R = \frac{a_R^2}{\sigma_R}$ and $\gamma = 1.4$.

- For both dimensional and nondimensional variables, $a^2 = \gamma \frac{p}{\rho} = \gamma p \sigma$, so that the equation form is the same.

Linearized Euler Equations (LEE) in 2D.

$$\frac{\partial \sigma}{\partial t} + u_b \frac{\partial \sigma}{\partial x} + v_b \frac{\partial \sigma}{\partial y} - \sigma_b \frac{\partial u}{\partial x} - \sigma_b \frac{\partial v}{\partial y} + u \frac{\partial \sigma_b}{\partial x} + v \frac{\partial \sigma_b}{\partial y} - \sigma \frac{\partial u_b}{\partial x} - \sigma \frac{\partial v_b}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + u_b \frac{\partial u}{\partial x} + v_b \frac{\partial u}{\partial y} + \sigma_b \frac{\partial p}{\partial x} + u \frac{\partial u_b}{\partial x} + v \frac{\partial u_b}{\partial y} + \sigma \frac{\partial p_b}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + u_b \frac{\partial v}{\partial x} + v_b \frac{\partial v}{\partial y} + \sigma_b \frac{\partial p}{\partial y} + u \frac{\partial v_b}{\partial x} + v \frac{\partial v_b}{\partial y} + \sigma \frac{\partial p_b}{\partial y} = 0,$$

$$\frac{\partial p}{\partial t} + u_b \frac{\partial p}{\partial x} + v_b \frac{\partial p}{\partial y} + \gamma p_b \frac{\partial u}{\partial x} + \gamma p_b \frac{\partial v}{\partial y} + u \frac{\partial p_b}{\partial x} + v \frac{\partial p_b}{\partial y} + \gamma p \frac{\partial u_b}{\partial x} + \gamma p \frac{\partial v_b}{\partial y} = 0.$$

Vector form of the LEE in 2D with a source.

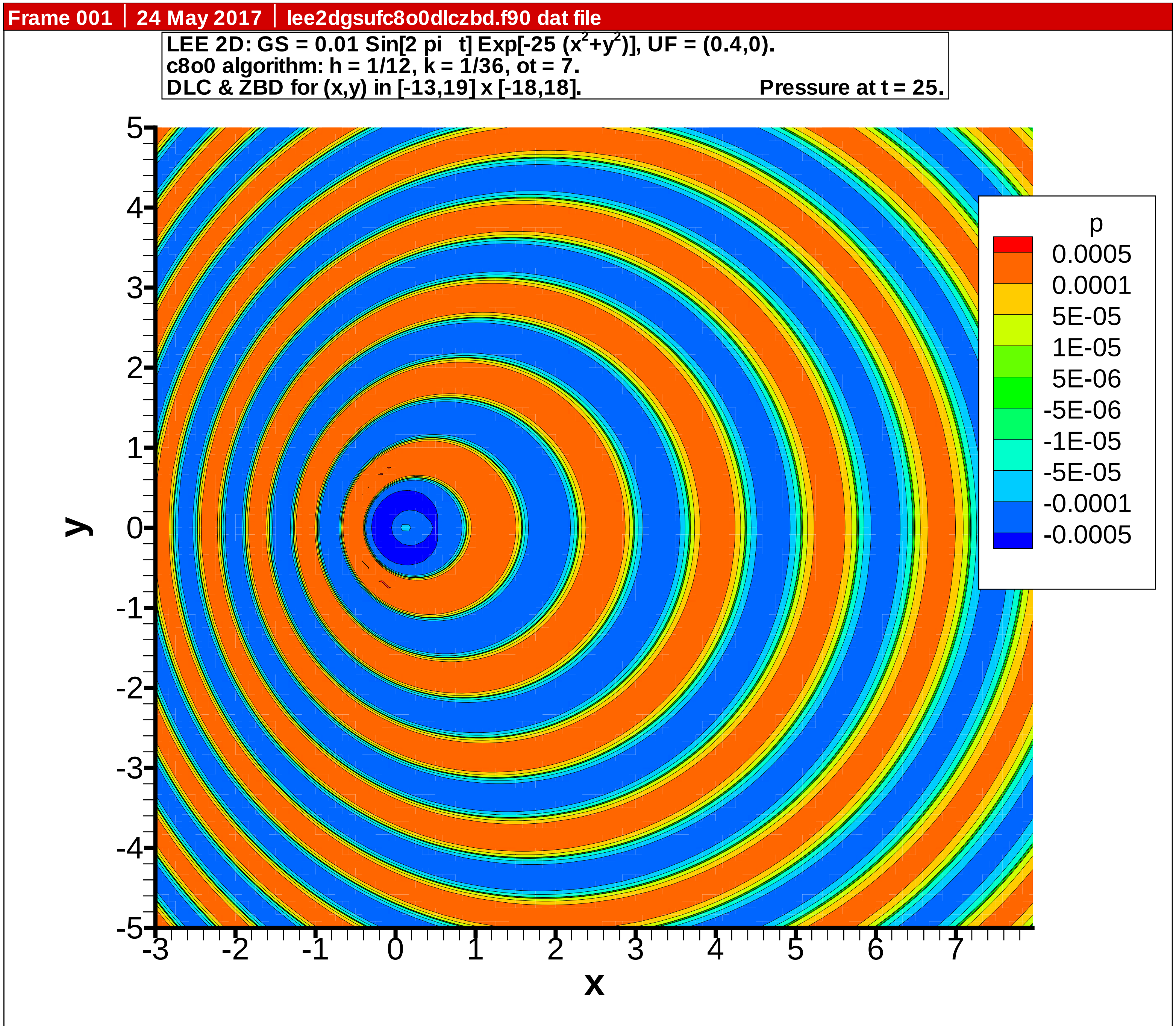
$$\frac{\partial \vec{V}}{\partial t} + A_b \frac{\partial \vec{V}}{\partial x} + B_b \frac{\partial \vec{V}}{\partial y} + A \frac{\partial \vec{V_b}}{\partial x} + B \frac{\partial \vec{V_b}}{\partial y} = \vec{S},$$

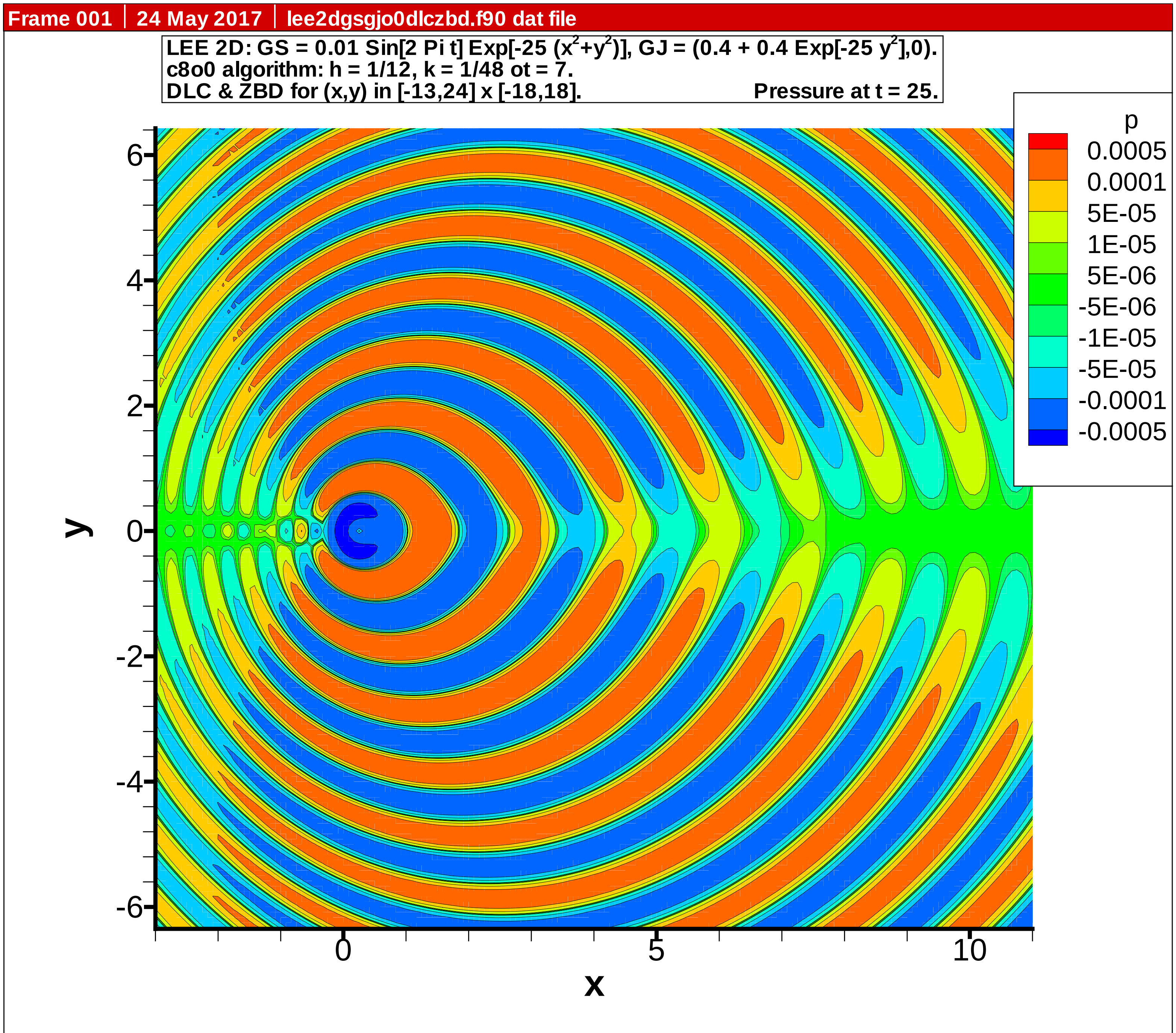
$$\vec{V} = (\sigma, u, v, p)^T = (1/\rho, u, v, p)^T,$$

$$\vec{S}(x, y, t) = (0, 0, 0, 0.01 \sin[2\pi t] \exp[-25(x^2 n + y^2)])^T.$$

- Either uniform base flow $\vec{V_b} = (1, 0.4, 0, 1/\gamma)^T$,
  or a parallel jet $\vec{V_b} = (1, 0.4 + 0.4 \exp[-25 y^2], 0, 1/\gamma)^T$.

- The coefficient matrices A and B are from $\vec{V}$,
  while the coefficient matrices A_b and B_b are from $\vec{V_b}$.


Frame 001
24 May 2017
lee2dgsufc8o0dlczbd.f90 dat file
LEE 2D: GS = 0.01 Sin[2 pi t] Exp[-25 (x²+y²)], UF = (0.4,0).
c8o0 algorithm: h = 1/12, k = 1/36, ot = 7.
DLC & ZBD for (x,y) in [-13,19] x [-18,18].
Pressure at t = 25.
p
0.0005
0.0001
5E-05
1E-05
5E-06
-5E-06
-1E-05
-5E-05
-0.0001
-0.0005
y
5
4
3
2
1
0
-1
-2
-3
-4
-5
x
-3
-2
-1
0
1
2
3
4
5
6
7


Frame 001 | 24 May 2017 | lee2dgsgjo0dlczbd.f90 dat file
LEE 2D: GS = 0.01 Sin[2 Pi t] Exp[-25 (x^2+y^2)], GJ = (0.4 + 0.4 Exp[-25 y^2],0).
c8o0 algorithm: h = 1/12, k = 1/48 ot = 7.
DLC & ZBD for (x,y) in [-13,24] x [-18,18].
Pressure at t = 25.
p
0.0005
0.0001
5E-05
1E-05
5E-06
-5E-06
-1E-05
-5E-05
-0.0001
-0.0005
y
x
6
4
2
0
-2
-4
-6
0
5
10

Damping Layers.

- The numerical domain is $\Omega_N = [-x_L, x_R] \times [-y_B, y_T]$.

- Unsteady pressure source p_S centered at the origin in Ω_N.

- Damping layers of width w_R, w_L and w_Y around Ω_N.

- The complete computational domain is
  $\Omega_C = [-(x_L + w_L), x_R + w_R] \times [-(y_B + w_Y), y_T + w_Y]$,
  with $\Omega_N \subset \Omega_C$ as the inner core where accuracy is desired.

N^{th} order polynomial damping profiles A.

Damping is done with polynomial damping profiles in x,

$$
\begin{aligned}
D_x(x) &= \delta_R((x - x_R)/w_R)^N, && \text{for } +x_R < x \leq +x_R + w_R, \\
&= 0, && \text{for } -x_L \leq x \leq +x_R, \\
&= \delta_L((x - x_L)/w_L)^N, && \text{for } -(x_L + w_L) \leq x < -x_L,
\end{aligned}
$$

and in y,

$$
\begin{aligned}
D_y(y) &= \delta_T((y - y_T)/w_Y)^N, && \text{for } +y_T < y \leq +y_T + w_Y, \\
&= 0, && \text{for } -y_B \leq y \leq +y_T, \\
&= \delta_B((y - y_B)/w_Y)^N, && \text{for } -(y_B + w_Y) \leq y < -y_B.
\end{aligned}
$$

N^{th} order polynomial damping profiles B.

- We assume $w_R = w_L = w_y = w$, with $w = 5$ or $w = 10$.

- We assume $\delta_R = \delta_L = \delta_y = \delta$, with $0 \le \delta \le 50$.

- We consider $N = 2$, $N = 4$, $N = 6$ and $N = 8$.

- Damping is done with modified governing equations in Ω_C,

$$\frac{\partial \vec{V}}{\partial t} + A_b \frac{\partial \vec{V}}{\partial x} + B_b \frac{\partial \vec{V}}{\partial y} + A \frac{\partial \vec{V}_b}{\partial x} + B \frac{\partial \vec{V}_b}{\partial y} + (D_x(x) + D_y(y))\vec{V} = \vec{S}.$$

Centered-Staggered/Cauchy-Kowaleskya/Tailor cno0 Propagation Method.

- Interpolation of an n^2 data surface by dimensional recursion.

- Cauchy-Kowaleskya recursion for time derivatives

$$
\begin{aligned}
\frac{\partial^{a+b+m}\partial_t \vec{V}}{\partial x^a \partial y^b \partial t^m} &= -\frac{\partial^{a+b+m}(A_b \partial_x \vec{V} + B_b \partial_y \vec{V})}{\partial x^a \partial y^b \partial t^m} \\
&\quad -\frac{\partial^{a+b+m}(A \partial_x \vec{V}_b + B \partial_y \vec{V}_b)}{\partial x^a \partial y^b \partial t^m} \\
&\quad -\frac{\partial^{a+b+m}(D_x + D_y)\vec{V}}{\partial x^a \partial y^b \partial t^m} + \frac{\partial^{a+b+m}\vec{S}}{\partial x^a \partial y^b \partial t^m}.
\end{aligned}
$$

- Time advance by Taylor series.

Maximum Relative Error with No Damping ($\delta = 0$): A.

- c4o0 computes with $h = 1/8$ and $k = 1/24$.
  c6o0, c8o0 and c10o0 compute with $h = 1/6$ and $k = 1/18$.

- $\Omega_N = [-3, 7] \times [-5, 5]$, $\vec{V}_b = (1, 0.4, 0, 1/\gamma)^T$.

- Reported errors are

$$E_{R,\infty} = \frac{\max\{|\vec{V}(\vec{X}) - \vec{V}^*(\vec{X})| : \vec{X} \text{ in } \Omega_N\}}{\max\{|\vec{V}(\vec{X})| : \vec{X} \text{ in } \Omega_N\}} = \frac{\|\vec{V} - \vec{V}^*\|_{\Omega_N,\infty}}{\|\vec{V}\|_{\Omega_N,\infty}},$$

  where $\vec{V}^*$ is a comparison solution from the same codes.

Maximum Relative Error with No Damping ($\delta = 0$): B.

Maximum relative error $E_{R,\infty}$ with $w = 5$ and no damping.

T	$c4o0$	$c6o0$	$c8o0$	$c10o0$
25	1.4930D-03	3.1436D-03	8.9365D-03	1.8767D-02
50	6.6326D-05	2.7791D-04	7.3701D-03	2.4321D-02
100	3.1044D-05	2.6507D-04	7.3527D-03	2.4521D-02
200	2.3663D-05	2.2449D-04	7.2923D-03	2.4517D-02
300	2.2813D-05	2.2534D-04	7.2911D-03	2.4516D-02

Maximum relative error $E_{R,\infty}$ with $w = 10$ and no damping.

T	$c4o0$	$c6o0$	$c8o0$	$c10o0$
50	5.2331D-04	9.1656D-04	2.3565D-03	1.0331D-02
100	2.6468D-05	4.4113D-05	1.9533D-03	1.0577D-02
200	1.9987D-06	1.2023D-05	1.9407D-03	1.0589D-02
300	7.9275D-07	1.0818D-05	1.9408D-03	******

Maximum Relative Error with No Damping ($\delta = 0$): C.

- Outer boundary error is propagated like any other signal.

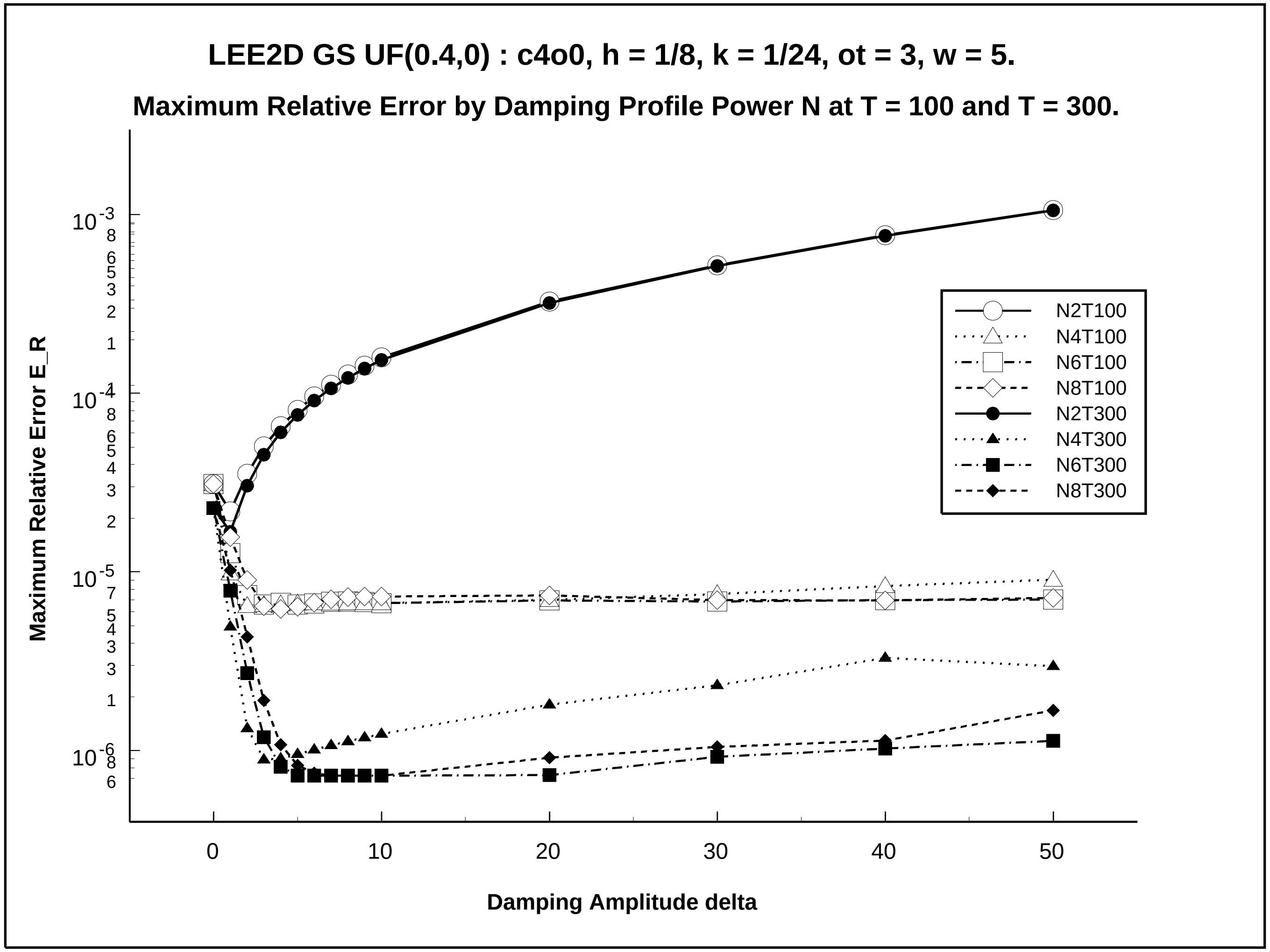
- $E_{R,\infty}$ increases with algorithm order, for each T and w, due to implicit damping from the greater diffusivity of the lower order algorithms.

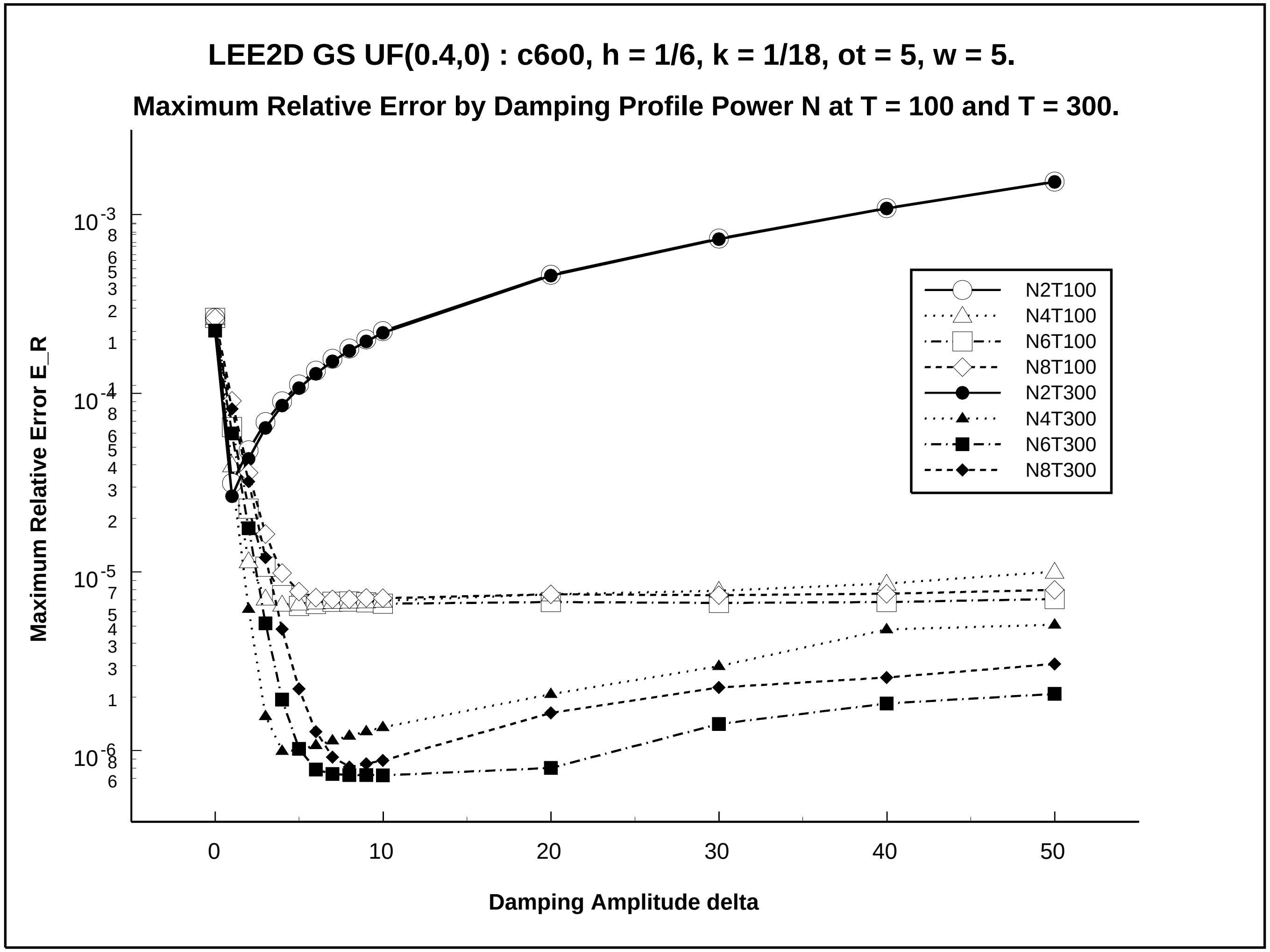
- $E_{R,\infty}$ generally decrease with simulation time T.

- A dissipitive dynamical systems with a periodic driving force converging to a periodic solution, any initial transient decays.

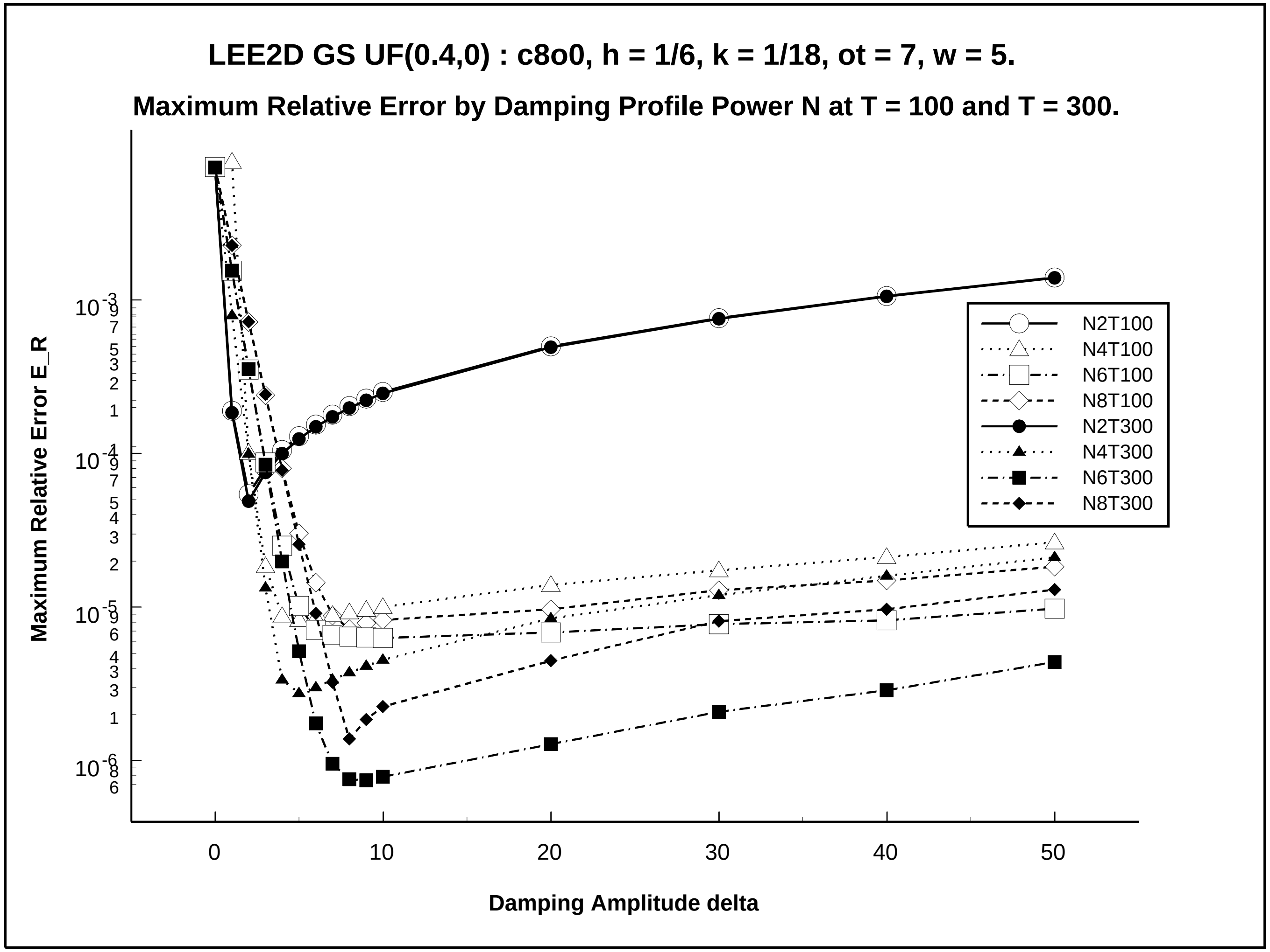
Damping With a Uniform Flow: A.

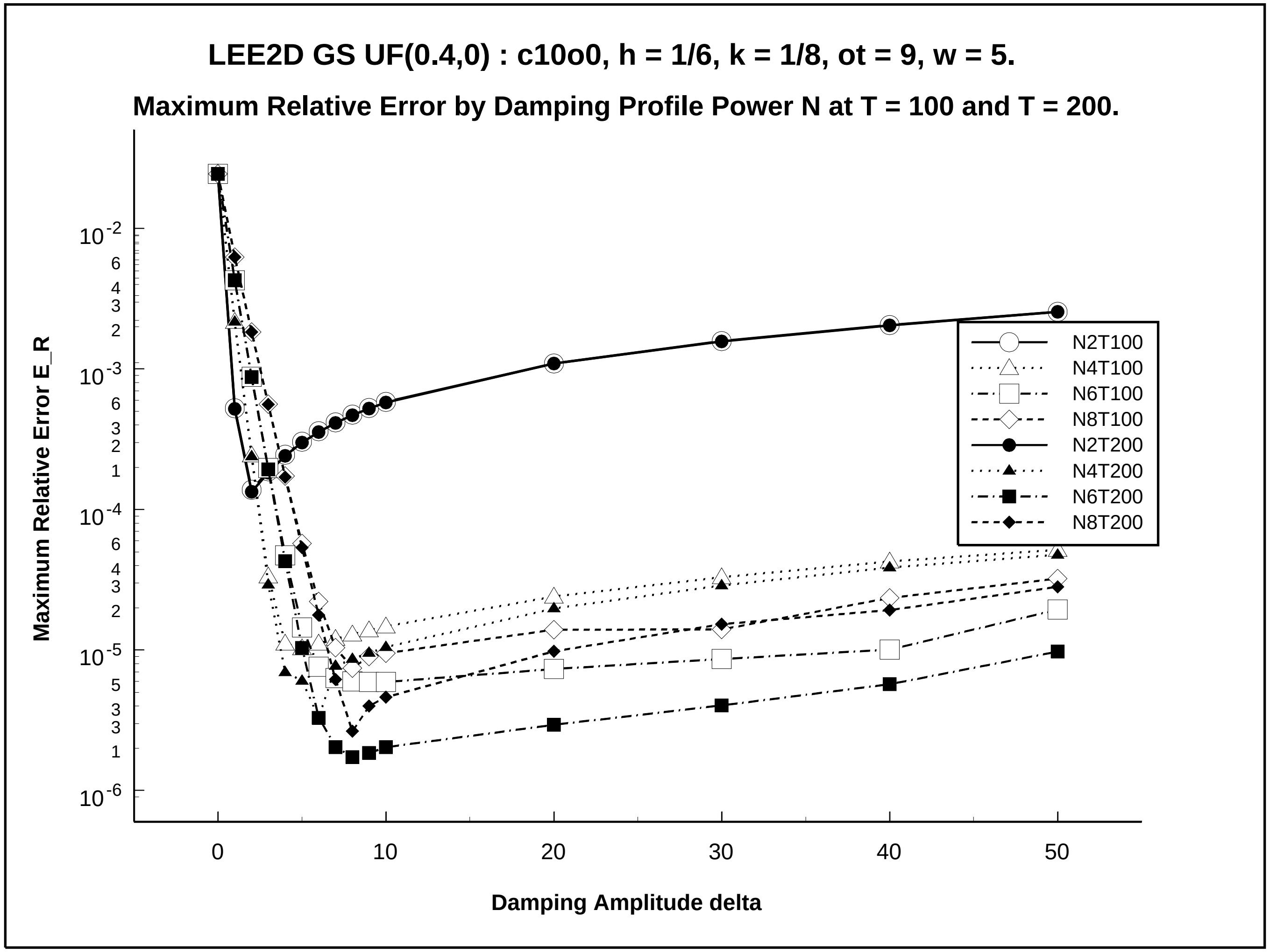
- Uniform flow $\vec{V}_b = (1, 0.4, 0, 1/\gamma)^T$ for (x, y) in $[-3, 7] \times [-5, 5]$.
- For c4o0, $\triangle x = \triangle y = h = 1/8$ and $\triangle t = k = 1/24$.
- For c6o0, c8o0 and c10o0, $h = 1/6$ and $k = 1/18$.
- Simulation time $T = 100$ or $T = 300$ (or $T = 200$ for c10o0).
- Damping layer widths $w = 5$ or $w = 10$.


LEE2D GS UF(0.4,0) : c4o0, h = 1/8, k = 1/24, ot = 3, w = 5.
Maximum Relative Error by Damping Profile Power N at T = 100 and T = 300.
Maximum Relative Error E_R
Damping Amplitude delta
N2T100
N4T100
N6T100
N8T100
N2T300
N4T300
N6T300
N8T300
0
10
20
30
40
50



LEE2D GS UF(0.4,0) : c6o0, h = 1/6, k = 1/18, ot = 5, w = 5.
Maximum Relative Error by Damping Profile Power N at T = 100 and T = 300.
Maximum Relative Error E_R
Damping Amplitude delta
N2T100
N4T100
N6T100
N8T100
N2T300
N4T300
N6T300
N8T300


LEE2D GS UF(0.4,0) : c8o0, h = 1/6, k = 1/18, ot = 7, w = 5.
Maximum Relative Error by Damping Profile Power N at T = 100 and T = 300.
Maximum Relative Error E_R
Damping Amplitude delta
N2T100
N4T100
N6T100
N8T100
N2T300
N4T300
N6T300
N8T300
0
10
20
30
40
50



LEE2D GS UF(0.4,0) : c10o0, h = 1/6, k = 1/8, ot = 9, w = 5.
Maximum Relative Error by Damping Profile Power N at T = 100 and T = 200.
Maximum Relative Error E_R
Damping Amplitude delta
N2T100
N4T100
N6T100
N8T100
N2T200
N4T200
N6T200
N8T200

Damping With a Uniform Flow: B.

- For $N = 2$, $E_R(\delta \geq 2) > E_R(\delta = 0)$,
  and $E_R(\delta = 50)$ is $O[10^2]$ greater than $E_R(\delta = 0)$.

- $N = 2$ can produce errors larger than not damping at all.

- For $N \neq 2$, $E_R(\delta \geq 2)$ is $O[10]$ to $O[10^2]$ less than $E_R(\delta = 0)$.

- N polynomial damping has an N^{th} order discontinuity at $\partial\Omega_N$.

- Quadratic damping more rapidly distorts the propagating solution surfaces as they enter the damping layer.

Damping With a Uniform Flow: C.

Smallest observed error $E_{R,\infty}$ with $w = 5$ at $T = 100$.

algorithm	$N = 2$	$N = 4$	$N = 6$	$N = 8$
c4o0	2.1854D-05	6.4663D-06	6.5709D-06	6.2108D-06
c6o0	3.1538D-05	6.5983D-06	6.4781D-06	7.0472D-06
c8o0	5.4334D-05	8.2486D-06	6.3638D-06	7.2109D-06
c10o0	1.3842D-04	1.0321D-05	5.9463D-06	7.4360D-06

Smallest observed error $E_{R,\infty}$ with $w = 5$ at $T = 300$.

algorithm	$N = 2$	$N = 4$	$N = 6$	$N = 8$
c4o0	1.6771D-05	8.9120D-07	7.2492D-07	7.2492D-07
c6o0	2.6601D-05	9.9872D-07	7.3028D-07	8.1127D-07
c8o0	4.8991D-05	2.7558D-06	7.4484D-07	1.3917D-06
c10o0*	1.3421D-04	3.2921D-06	1.7333D-06	2.6527D-06

c10o0* at $T = 200$.

Damping With a Uniform Flow: D.

- $O[10^{-4}] \le E_R(T = 100) \le O[10^{-6}]$,
  $O[10^{-4}] \le E_R(T = 300) \le O[10^{-7}]$.

- The upper right section has the lowest levels in each table.

- Higher N works better with all of the algorithms,
  with more necessary improvement for higher order algorithms.

- As a rule of thumb, the damping profile power should be at least as large as the algorithm order.

- The 9^{th} order c10o0 might need either $N \ge 10$ or $T \ge 300$.

The Effect of Damping Layer Width: A.

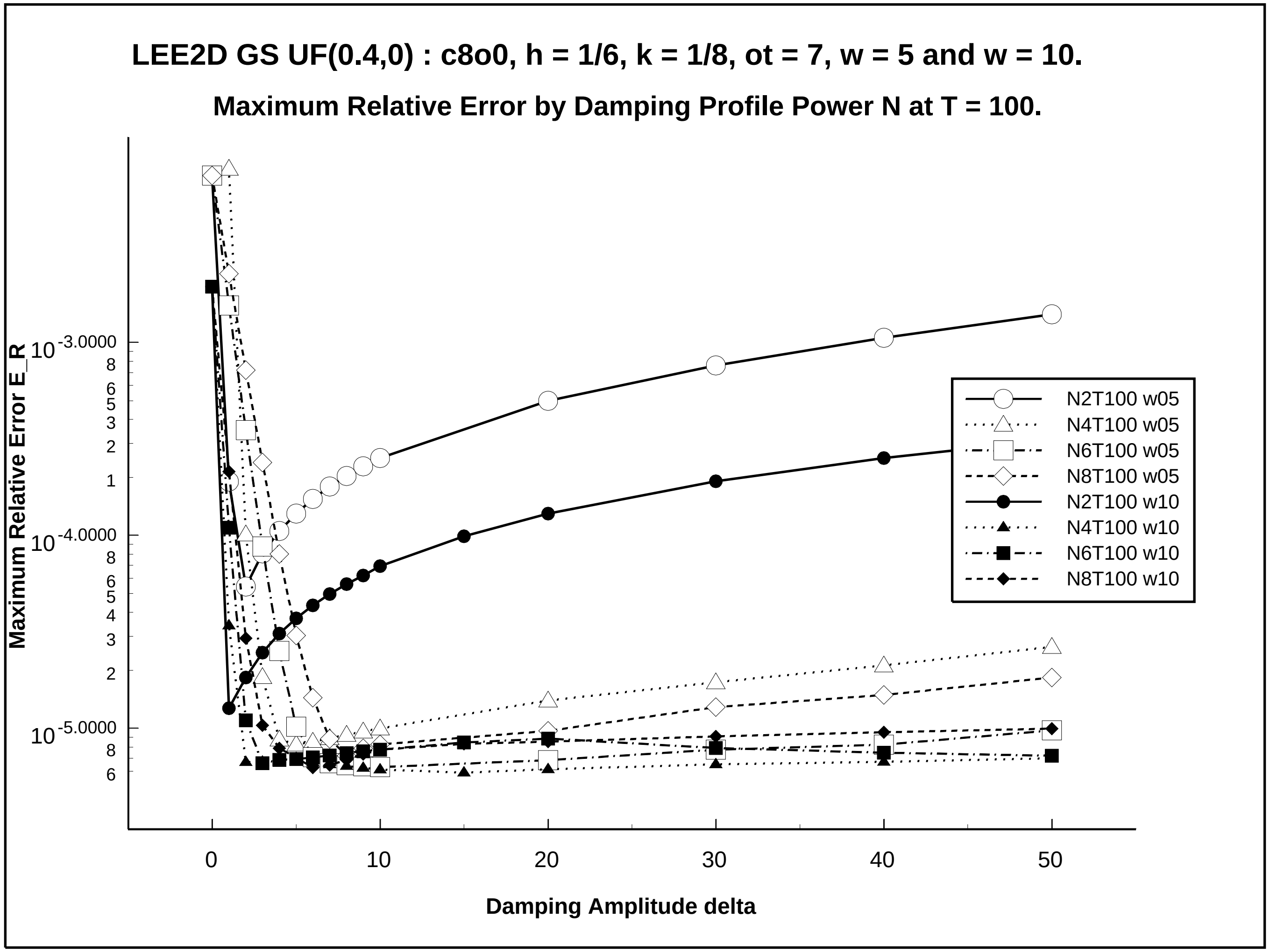
- Uniform flow $\vec{V}_b = (1, 0.4, 0, 1/\gamma)^T$ for (x, y) in $[-3, 7]\times[-5, 5]$.

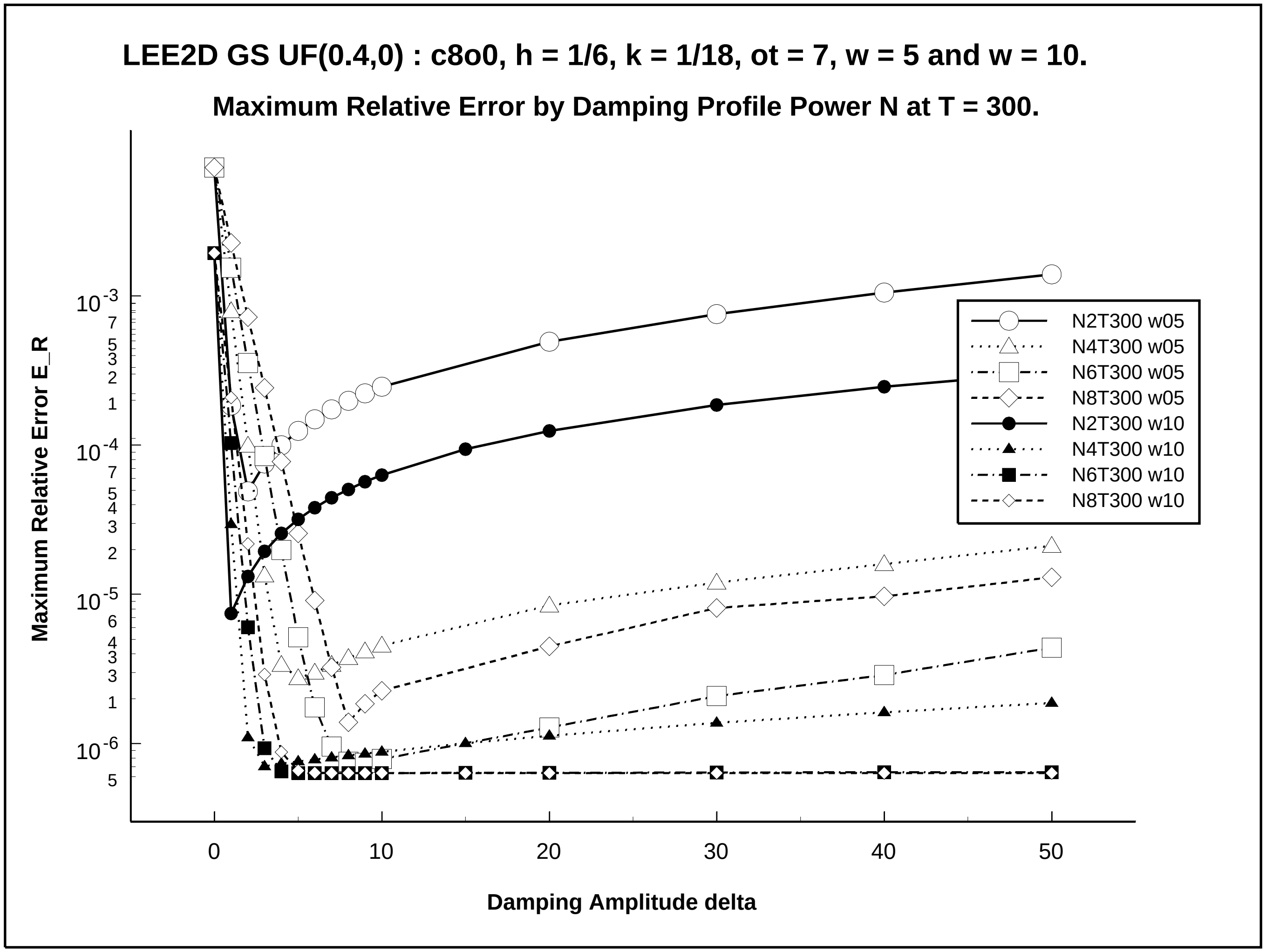
- c8o0 algorithm with $h = 1/8$ and $k = 1/24$.

- Damping layer widths $w = 5$ and $w = 10$.

- Simulation times $T = 100$ and $T = 300$.

-


LEE2D GS UF(0.4,0) : c8o0, h = 1/6, k = 1/8, ot = 7, w = 5 and w = 10.
Maximum Relative Error by Damping Profile Power N at T = 100.
Maximum Relative Error E_R
Damping Amplitude delta
N2T100 w05
N4T100 w05
N6T100 w05
N8T100 w05
N2T100 w10
N4T100 w10
N6T100 w10
N8T100 w10



LEE2D GS UF(0.4,0) : c8o0, h = 1/6, k = 1/18, ot = 7, w = 5 and w = 10.
Maximum Relative Error by Damping Profile Power N at T = 300.
Maximum Relative Error E_R
Damping Amplitude delta
N2T300 w05
N4T300 w05
N6T300 w05
N8T300 w05
N2T300 w10
N4T300 w10
N6T300 w10
N8T300 w10

The Effect of Damping Layer Width: B.

- Error reduction by doubling w is greater for lower N.

- Error reduction by tripling T is greater for higher N.

- For $N = 2$, error decrease is more efficient by increasing w, but the error is generally the largest if all else is the same.

- For $N \neq 2$, error decrease is more efficient by increasing T, and most efficient by increasing both w and T.

Damping With a Parallel Jet: A.

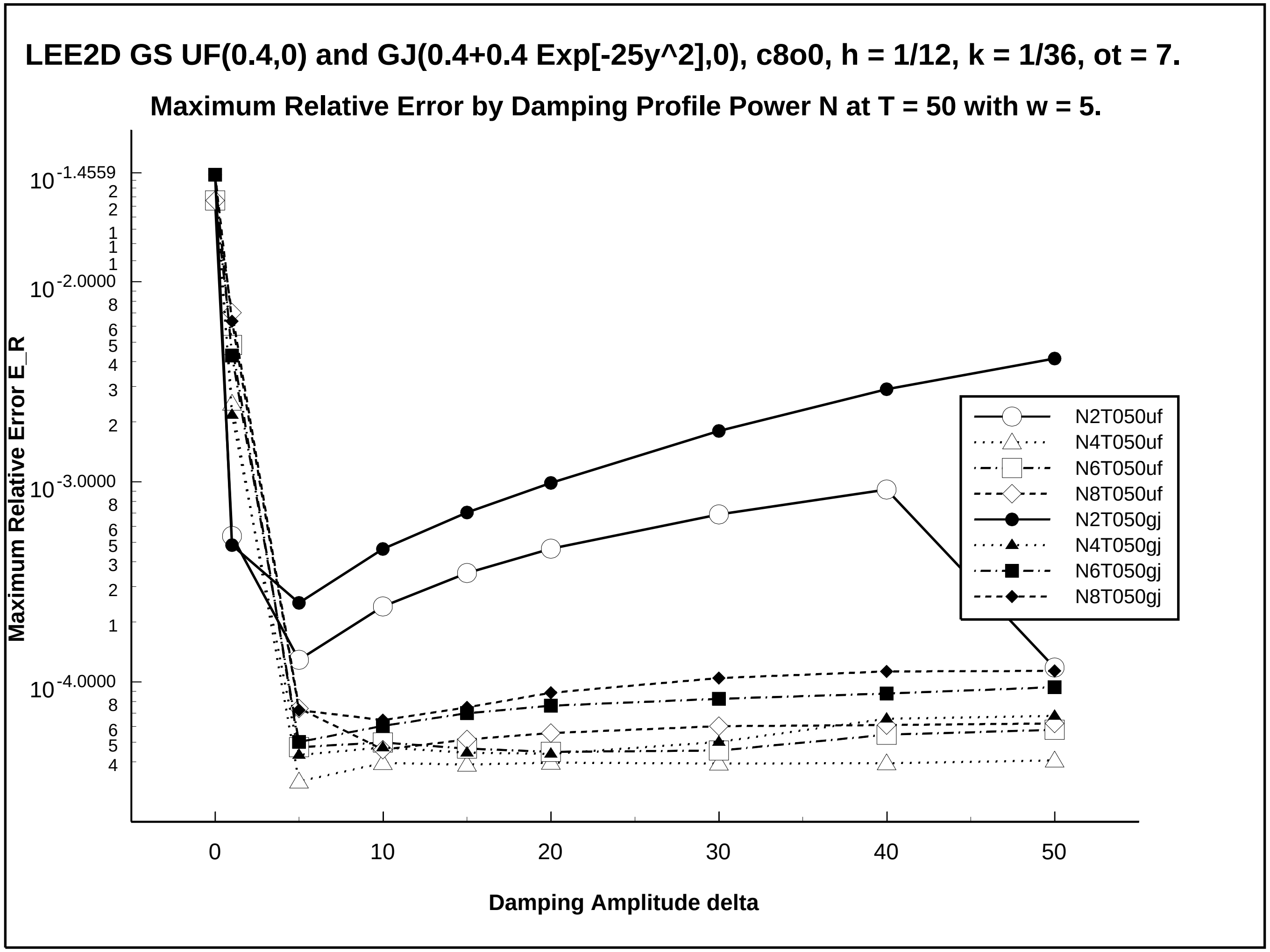
- $\vec{V_b} = (1, 0.4+).4\,\mathsf{exp}[-25y^2], 0, 1/\gamma)^T$ **OR** $\vec{\mathsf{V}_\mathsf{b}} = (1, 0.4, 0, 1/\gamma)$.

- $\Omega_N = [-3, 9] \times [-5, 5]$ **OR** $\Omega_\mathsf{N} = [-3, 7] \times [-5, 5]$.

- For c8o0 with $h = 1/12$ and $k = 1/36$.

- Simulation time $T = 50$.

- Damping layer width $w = 5$.



LEE2D GS UF(0.4,0) and GJ(0.4+0.4 Exp[-25y^2],0), c8o0, h = 1/12, k = 1/36, ot = 7.
Maximum Relative Error by Damping Profile Power N at T = 50 with w = 5.
Maximum Relative Error E_R
Damping Amplitude delta
N2T050uf
N4T050uf
N6T050uf
N8T050uf
N2T050gj
N4T050gj
N6T050gj
N8T050gj

Damping With a Parallel Jet: B.

- This data is for the relatively short simulation time $T = 50$.

- This grid resolution is doubled from the uniform flow simulations.

- Errors from the uniform and jet flow differ by ≈ 2 to ≈ 5, so the boundary treatment performs similarly for both.

- All of the effects of damping profile power, damping layer width, simulation time and algorithm order should apply to variable coefficient jet simulations with these algorithms just as they did to constant coefficient uniform flow simulations.

Conclusions.

- Error can decreases as simulation time T increases, from decaying transients.

- Error tends to decrease as the damping layer width increases, more effectively and efficiently for smaller N.

- Error tends to decrease as damping power N increases, more effectively and efficiently for larger N.

- As a rule of thumb, the damping profile power N should be at least as large as the algorithm order.

- Quadratic damping produces the largest errors.
- Damping with a jet and a uniform base flow are similar.
- Dissipitive algorithms implicitly dampen.
- High order algorithms propagate accurately,
  both intended signals and also errors.

Acknowledgments.

- The Commercial Supersonics Technology Project under NASA Advanced Air Vehicles Program supported this work.

- This work continues what was done in J. W. Goodrich, "Experiments in numerical error control for sound propagation in a 2D jet using a damping layer and nonreflecting outer boundaries," *Int. J. Aeroacoustics* **14** (2015), pp489-520.